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A LAST WORD FOR "THE QUADRATEURS"

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A LAST WORD FOR "THE QUADRATEURS"

A PAPER

READ FOR

"THE CLUB" (LITERARY)

— OF —

SPRINGFIELD, MASS.

APRIL 27TH, 1900

— BY —

A. R. BUFFINGTON.

(PRINTED BY REQUEST OF THE CLUB)

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(CIRCLE SQUARERS)

As long ago as 430 B. C., and as late as A. D. 1897 the "Quadrateurs" have been the butt of wits and the objects of pity, or contempt, for both the learned and unlearned, the world over.

About the year 430 B. C., Aristophanes, in "The Comedy of the Clouds," held Meton and Socrates up for ridicule, as "circle squarers." Anno Domini 1897 Professor John Fiske cannot end a biting criticism of the rectifiers of Shakespeare without a fling at the rectifiers of the circle.

Between these widely separated times, more or less evenly distributed over the long interval, undeterred by the want of faith of the mass of mankind, men have worked at the famous problem of "squaring the circle." Some knowing what they were doing and equipped for the work, some—many—without knowing, and not at all equipped; more's the pity. Good men and learned have quarreled over it, and others—from the simple minded, who rolled a wheel over the ground and crudely measured its circumference, to the still less wise, who felt an inspiration to do the work, discovering the solution in mysterious numbers and relations fortified by Holy Writ—have wasted both substance and happiness in a vain endeavor to solve the problem.

Mathematical giants — Newton and Leibnitz for instance—have unsuccessfully wrestled with it. Where these have failed, it would seem little short of madness for the ordinary man to attempt the solution. Newton pronounced the indefinite quadrature of the circle impossible. The more definite problem, the quadra-

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ture of the circle as a whole, has been solved, it is believed, these many years, within limits so small that the solution has been accepted as practically correct.

As the area of a circle depends on the length of its circumference it follows that were the ratio of the diameter—made the unit—to the corresponding circumference known, the circumference of all circles would become known. This ratio, called *pi*, the name of the Greek letter π, used in mathematics to express it, has been determined approximately, by determining the lengths of the peripheries of circumscribed and inscribed polygons of many sides. A circle's circumference must be greater than the periphery of any inscribed polygon and less than that of any circumscribed. Increasing the number of sides, increases the length of the periphery of the former and decreases that of the latter, and when the two become to many places of decimals identical, it would seem,—the circle being between them,—that, to that extent at least, the length of the circle's circumference has been obtained. This multiplied by half the radius—that is, by ¼—the area of the circle, the diameter of which is the unit, is obtained, and thus the circle "squared." My knowledge of the history of the efforts to "square the circle" in the above statements and in what follows, is obtained at second hand, from the introduction to "The Quadrature of the Cir- "cle, The Square Root of Two, and The Right-Angled "Triangle" by Wm. Alexander Myers, professor of Myers' Commercial College, Louisville, Ky., 1873, which introduction (of only 64 closely printed pages!) begins with a translation by I. Babin, A. B., of Louisville, Ky., of the history of the Quadrature of the Circle by Montaclu.

This introduction is followed by "Part first, con- "taining the Geometrical and Final Solution of the "Quadrature of the Circle, by an entirely new method, "together with ample proofs of the same." The method is certainly "new" and the proofs "ample," in the appalling array of figures, magnificent but devoid of mathematical basis. The first step being conceded, the remainder, so gorgeously furnished with mathematical trappings, would follow as a matter of course.

The extravagant expenditure of *figures* in this "final solution," is a fitting climax for the incalculable loss of mental energy wasted on this problem from the time of Socrates to the present.

Twenty years later—1893—Rufus Fuller, Boston, Mass., announces in a pamphlet, "printed for the author," of 31 pages, suitably prefaced, and appendixed by numerous diagrams, rich in geometrical ornamentation, ("Not found in any ancient or modern works,") "A Double Discovery. Square of the Circle. The "long sought for absolute fraction found in place of "incorrect, approximate fractions and in place of unend-"ing decimals."

"The famous problem of squaring the circle is solved "in Geometry."

"In the mathematical part, in place of unending "decimals there is shown the absolute fraction now "discovered, that was searched for by the ancient "Greeks and the ancient problem is solved in mathe-"matics."

"Both parts taken together constitute the Double "Discovery"—*to find* which is the problem, (one more difficult than squaring the circle itself), that confronts the searcher after truth should he venture into the wonderful maze by which this latest "squarer" arrives at his conclusions.

"In the explanations algebraic signs are not used. "but plain terms; and coincidently the solution of this "most widely known of the intricate problems, was "effected during the four hundredth, circling, quad-"rangular year, 1892"!

"The rounded year 1000 being also a year of an "ancient historical event even in the Eastern part of "the Western half of the world, it is alluded to in con-"nection with the cut of a lofty tower at the end of the "diagrams," and he accordingly ends his pamphlet by a picture of the "Norse Tower," and, after a description of which, he adds, "This part of the present work

" also ends with recent adjudged discoveries showing " the location of the Caucasian race in the New World " A. D. 1000 " !

What these two squarers, with "*final*" solutions, would meet with at the hands of Montaclu may be conjectured after a perusal of his history, so replete with quaint humor, often of a most biting character.

This history could not be condensed, being made up of short, condensed, notices of the many "squarers" from Socrates to the year A. D. 1801. To be fully understood and appreciated it should be read as a whole, but an extract or two may serve to convey some notion of its character, the character of the material dealt with, and serve the purpose of this paper. It begins with the statement, " It has seemed to us fit to treat " here separately this Great Question on account of its " great celebrity, and we shall not hesitate to give some " of the silly notions to which this problem has given " rise in ill-balanced and enthusiastic minds."

Respecting the solution of the problem itself this emphatic and unqualified statement is made, viz.:

" But this no one has yet been able to do, nor is it " likely ever to be done."

This history shows that Montaclu and the French Academy of Sciences arrived at the conclusion that Archimedes had, by his work, fixed forever the limits beyond which the ratio of the diameter to the circumference could not go. Archimedes, this history says, " showed about the year 250 B. C. that if the diameter " of a circle is 1, its circumference is less than 3 10-70 " or 3 1-7, and more than 3 10-71 ; by taking 3 1-7 the " error is less than the 1-497 of the diameter. The cal" culation of Archimedes is singularly skillful and antici" pates the objection made by some of those who reject " his account for the reason that he could not precisely " extract the square roots of the several numbers used " in his calculation. But I have known some of these " individuals, and I have never found a single one who " knew Archimedes otherwise than *by name*."

" We pass rapidly over the centuries of ignorance

" which produced a few treatises on the quadrature of " the circle, manuscripts left remaining in the dust of " libraries, until we reach the period of the revival of " letters among ourselves. About this time the famous " Cardinal de Cusa distinguished himself by his unfor- " tunate attempts at the solution of this problem. Nev- " ertheless he tried an ingenious method ; he rolled a " circle on a plane or line, and supposing that its cir- " cumference was applied to it wholly until the point " which had first touched it touched it again ; he there- " fore justly inferred that this line would be equal to " the circumference. He even conceived the outline of " the curve, which the point that first touched the " straight line was to describe which formed the curve, " since called the Cycloid. * * * He also " tried another method, according to which he gave the " following solution of the problem : a circle being " given, add to its radius the side of the inscribed square, " and with this line as diameter describe a circle, in " which is inscribed an equilateral triangle, the perim- " eter of this triangle will, says Cardinal de Cusa, be " equal to the circumference of the first circle."

" It was not difficult for Regiomontanus to prove " that Cusa was mistaken. * * * Besides, " the Cardinal, learned for his age, though very much " addicted to astrology, presents in the collection of " his works several geometrical tracts which are full of " paralogisms."

" We have just spoken of Charles de Bovelle * * " distinguished at the time by the title of 'Noble Phil- " osopher.' He signalized himself by the strangest " ideas. He gave in 1507 a work entitled : *Introductio- " num Geometricum.* * * * He claims to give " there the quadrature of the circle according to the " idea of the Cardinal de Cusa, which, he says, came to " him by seeing a wheel moving on the pavement. But " the construction by which he pretends to give the " length of the line to which is applied the circumfer- " ence of the rolling circle is absolutely arbitrary, and " it would follow that the diameter is to the circumfer-

"ence as 1 is to the square root of 10, or 3.1618, which "is far from the limits of Archimedes. What is also "singular is, that in this same book, and in an appendix "added to the first volume of the preceding work, he "speaks of the quadrature of the circle made by a poor "peasant, according to which the circle having 8 for "diameter is equal to the square having 10 for diagonal, "that is to 50, which is false; for the circle is in this "case less than 50 2-7, and more than 50 18-71, and the "quadrature of Bovelle does not agree with that of the "peasant, which he considers as true; for the latter "gives the relation of the diameter to the circumference exactly as 1.0000 to 3.1250; the 'Noble Philosopher' even wanders further from the truth than the "peasant does; and he might have been told that when "one is mistaken he ought not at least to contradict "himself. Bovelle says, falsely, that these relations "coincide. * * *"

"The Spanyard, Sir Jaime Falcon, of the order of "Our Lady of Montesa, published in 1587, at Antwerp, "his paralogism on the quadrature of the circle. His "book is rendered amusing by a dialogue in verse "between himself and the circle, which thanks him "very affectionately for squaring it; but the good and "model knight ascribes all the honor thereof to the "holy patroness of his order. * * *"

"But a much more famous than the foregoing challenged the attention of learned Europe by his pretensions on the quadrature of the circle; it was the celebrated Joseph Scaliger. Full of self-conceit, he supposed that he had only to present himself on the field "of geometry and that nothing that had baffled geometricians until then could resist a man of letters "with his powers. He therefore undertook to find the "quadrature of the circle, and put forward, with much "braggadocia, his discoveries on this subject in a book "which appeared in 1592: *Nova Cyclometria;* but he "had no cause to congratulate himself on having thus "wished to place himself among geometricians. For he "was refuted by Clavius, Viete, Adrianus Romanus,

" Christman, etc., who showed, each in his own way, " that the size which he assigned to the circumference " of the circle was only a little less than the inscribed " polygon of 192 sides; which being absurd, demon-" strated the incorrectness of Scaliger's reasoning; but " he did not surrender; and never did a man who " thought he had discovered the quadrature of the cir-" cle, the trisection of the angle, the duplication of the " cube, or perpetual motion, give in to the plainest rea-" soning. He will sooner deny the most elementary " propositions of geometry, like Moliensis Cana, who " found no less than twenty-seven false propositions in " the first book of Euclid. Scaliger replied with bitter-" ness to the geometricians who had censured his quad-" rature; he treated them with contempt, especially " Clavius, who had already wounded him by an answer " to his attack on the Gregorian Calendar. Unfortun-" ately for the honor of Scaliger abuse is not reasoning, " and the established fact remains that Scaliger, an " eagle in literature, was nothing in geometry."

" Scarcely had Scaliger disappeared when one " Thomas Gephirander came to take his place. But he " had not Scaliger's pride; he acknowledged, even in " the title of his book, that his discovery was simply " the result of Divine Grace. We shall see many others " gifted with this same spirit of humility. * * * " But scarcely any of the follies with which false rea-" soning, a false mind, and the conceit of never recant-" ing one's errors inspire these visionaries have equaled " those of Alph. Cano, of Molina. * * * He " remodels the whole of Euclid, and scarcely one of his " propositions is spared by him. Yet who would believe " it! he found another fool, named Janson or Jansen, " who translated him into Latin under the title of *Nova* " *reperta Geometrica*, etc. Morover Cano admitted that " he had not the least idea of geometry until the Deity, " whose delight it is to humble the proud and enlighten " the ignorant, had inspired him."

" A similar wiseacre presented at the same time in " France his paralogisms upon the quadrature of the "circle and the duplication of the cube. It was a mer-"chant of Rochelle, called De Laleu. This one also "pretended to have received the solution of these prob-"lems by *divine revelation*, and announced that the "union of the Jews, Turks, and Pagans to the Christian "religion depended upon the manifestation of this "truth. In fact, according to him, the quadrature of "the circle was the quadrature of the heavenly Tem-"ple, and the duplication of the cube that of the ele-"mentary, terrestrial, and aquatic altar, whence was to "flow the conversion of the Jews, idolaters, etc. * *"

"An anonymous writer proclaimed in 1671 that the "reign of the greatest king of the universe was to be "rendered illustrious by this most brilliant discovery, "and undertook to prove it by a pamphlet in 4to, enti-"tled: Demonstration of the Divine Theorem of the "Quadrature of the Circle, and the relation of this the-"orem with the Visions of Ezechiel and the Revelation "of St. John. He does not fail, after the example of "his colaborers, to ascribe his discovery to a special "favor of the Divinity. * * * In fact, there "is found at the end of this work a large mysterious "board, presenting on a common center four decreasing "pyramids of circles and angles, which represent the "Angelic Hierarchy."

"The third fool was named Dethlef Cluver, grand-"son or nephew of the celebrated geographer of that "name. By ransacking the science of the infinite, on "which he promised a great treatise, he finitely discov-"ered that this problem, to find the quadrature of the "circle, reduces itself to this one: to construct a world "analogous to the divine intelligence. * * *"

"But among all the discoverers of the quadrature of "the circle one is a kind of phenomenon, the only one "who as yet admitted his mistake. It is Richard "(White), an English Jesuit. * * * But some "friends opened his eyes, and he * * acknowledged "his error. * * *"

"A better knowledge of geometry did not keep the "18th century from similar follies; there is not even a "doubt that succeeding ages will resemble in this "respect the past. * * *"

"The solution of the three problems which have so "long puzzled narrow intellects, the quadrature of the "circle, perpetual motion, and the trisection of the "angle was also announced in 1714 with much emphasis. * * *"

"Usually the (*quadrateurs*) come off with seeing "their discoveries neglected or scoffed at by their contemporaries; but in 1728, Mathulon, of Lyons, was "more unfortunate. He announced to the learned world "his signal discoveries of the quadrature of the circle "and perpetual motion. He was so confident of his "success that he appropriated 1,000 crowns for whoever "would demonstrate to him that he was mistaken on "either of these points."

"In 1753 an officer in the guards, Sir de Causans, "who until then never had any suspicion of geometry, "suddenly found the quadrature of the circle in having "a circular piece of sod cut, and then, rising from truth "to truth, explains by his quadrature original sin and "the Trinity."

"We must not forget one of the modern *Quadrateurs* who outdid many others in sanguineness and "absurdity. It is Rohberger de Vausenville. The challenges he had made to the geometricians of all nations, "even Turk and Arabian, as well as to all academies, "the suit brought forward against the Academy of Sciences to secure for himself the capital prize established by Count de Meslay, his indecent attacks upon "all geometricians who tried to enlighten or teach him, "have made him famous among those who have followed this path."

"We have also just received a print with the title: "Final Solution of the Diameter of the Circle to its "Circumference, or the discovery of the Quadrature of "the Circle, by Christian Lowenstein, Architect, Cologne, 1801."

"These publications come to us more especially in "the Spring of the year, when fits of folly are more "frequent, and *cit. de la Land*, who spent a year at "Berlin, says it was the season when the Academy of "Berlin received most writings of that kind."

"One is disposed to think this quadrature impossi-"ble to the human intellect when the useless efforts of "geometricians of all times are considered ; I do not "speak of the pitiable efforts of those we have just "been discussing, but of the efforts of such modern "geometricians as St. Vincent, Wallis, Newton, Leib-"nitz, Bernoulli, Euler, etc., who have found new meth-"ods of determining the area of curves, and who, by "their reasoning, have found that of a quantity of other "curves less complicated in appearance than the circle, "whereas the latter has always eluded their efforts."

"Besides, a distinction must be made in this respect : "there are two quadratures of the circle, one definite, "the other indefinite. The definite quadrature is the "one that would give the precise measure of the entire "circle or of a given sector or segment, without giving "indefinitely that of any sector or segment whatever. "The indefinite quadrature, which would be the most "perfect by giving the quadrature of any part what-"ever, would evidently include the other. Scarcely any "but the first is sought by *quadrateurs* in general."

"The conviction is general that there is no demon-"stration absolutely convincing that the definite quad-"rature is impossible. Yet James Gregory claimed "that he gave an irrefragable demonstration."

"It rested upon the progressive course represented "by the increase and decrease of the inscribed and cir-"cumscribed polygons whose limit is the circle itself. "But this demonstration did not appear conclusive to "Huygens, and it was the cause of a contest between "these two geometricians which occupied the newspa-"pers of the time. It must be admitted that though "the reasoning is worthy of a head like that of Greg-"ory, one of the forerunners of Newton, yet as the last "limit of which he speaks is placed, so to speak, in the "mists of the infinite, the mind is not struck by an irre-"sistible conviction."

Let us consider the method referred to, viz : "The "increase and decrease of the inscribed and circum-"scribed polygons whose limit is the circle," — the method, I believe, relied upon to give the correct value of π, and which does give the value now universally accepted and used, and which is considered beyond the shadow of question. It would be, if the values found for the lengths of the perimeters of the polygons were correct and if a polygon could ever become a circle. A polygon so long as it has any sides at all cannot be a circle. That two polygons can include between them a circle is correct so long as the sides of the circumscribed are tangent to the circle and the sides of the inscribed have, each, two points in common with the circle. Any other position of circumscribed and inscribed polygons with reference to a circle is inconceivable. A mathematical line, whether it be straight or curved, is defined to be without breadth or thickness—has length only—and a mathematical point has neither length, breadth, nor thickness. The inscribed polygon can have but two points of each side common with the circle, and the circumscribed but one. Beyond these points neither can get into the circle's place, as defined by its circumference. Therefore neither, so long as they are figures—polygons—with sides, can represent the circle—equal in length the circle's circumference: one must always be less and the other greater. Were it possible to determine perfectly the lengths of the sum of the sides of these polygons, after such an increase in the number of these sides that the values throughout would be the same, one of two things would happen, either the inscribed would get out of the circle or the circumscribed would get into it ; either this, or they both would cease to be polygons and become one and the same circle, coincident with the circle, in which and about which, they were originally located. Otherwise we should have a new definition for a circle. It would be a polygon of an infinite number of sides, not a curve every point of which, having neither length, breadth, nor thickness, is equidistant from a like central point.

It is needless to say that the exact lengths of the perimeters of these polygons, even if the number of sides could be so increased, cannot be determined. Now what results: an approximate value is used, but this has two elements of error, arising in the first instance from the use of but part of a value, the *whole* of which is too small, though granted to be exceedingly so; and in the second, from the impossibility of getting this imperfect whole, to do which would require more years than there are miles between the Earth and the Sun. These elements of error will be brought out clearly by going a little into detail, as follows. The starting point in the determination of the perimeter of the inscribed polygon is all right. A six-sided polygon is the starting point, the length of each side of which, being exactly equal to the radius of the circle, and this radius being ½ the unit, the whole perimeter is in length exactly 3. The method pursued in increasing indefinitely the number of sides of the polygon, is to double them successively, for which there is a well-known formula. In the first operation of doubling the sides, the square root has to be taken of an imperfect square. The operation of taking this square root can be continued forever without arriving at a completed value, but must be cut short at some practicable number of decimals for subsequent operations. Therefore in the second doubling of sides, the start is made with an imperfect value—a value too small, being but part of a too small value. Hence a second loss must occur, and so on for every subsequent operation, and in the initial, and every subsequent operation, the loss is the loss in one side multiplied by the number of sides. These losses are admittedly very small, but each operation entails an ever increasing one, so that the sum total of loss must be appreciable. Thus it does not take the mind of a Leibnitz or a Newton to see that when the mathematician gets through with finding his value for the length of the perimeter of the inscribed polygon of any possible number of sides he has a value decidedly too small. Now in like manner he arrives at a value for the perimeter of the final circumscribed polygon, also too small, and when these two values agree to the extent of many places of decimals he decides he has the value of the length of the circle to that extent because he has been carrying all along the belief that the circle has remained between the two polygons, whereas the circle has escaped the net. It, the circle, has kept its length unimpaired while the net has become so imperfect that before the operator makes the final throw for the circle it has already escaped—the inscribed and circumscribed polygons, being too small from the very start, when—one extended and the other reduced—their length values become limitedly identical, no longer have the circle between them, and therefore cannot in any sense represent it. On such imperfect work as this, the weakest of any I know in the whole field of Mathematics, has been established a value that has become, through long accepted use, a kind of sacred fetich. To question this sanctified π — equivalent to self-destruction — calls down at once on the impious heretic universal condemnation. Whatever values for the lengths of the perimeters of the inscribed and circumscribed polygons, finite man may work out, though they coincide to many hundred places of decimals, the length of the perimeter of the inscribed must be always less than that of the circumscribed, just by so much as the difference between the remaining decimals of the values to infinity, the impossibility of finding and using which have been pointed out, and necessarily less than the circle itself. Therefore the value taken for π, being the whole number and coinciding decimals of the two values found, and therefore, so far as it goes but the value of the length of the perimeter of the inscribed polygon, is too small. The whole argument may be summed up in these few words. The operation of finding the values of two quantities, one greater and one less than a third, must be so conducted if they are to remain the limits of this third that the successive steps make no diminution. For if each step correspondingly diminishes both, both growing constantly less than their true values, the final values determined are not correct, and are not the limits to the value of the third, inside of which they have shrunk in the operation of increasing the smaller and diminishing the larger until a certain coincidence is found. In the creed of Mathematics the mythical quantity 3.14159+ is the kernel, and forms the test of orthodoxy, indeed sanity, of the would-be disciple. He must subscribe to that under penalty, not only of exclusion from the well beloved, but from the company of all the sane! The Doctors in the pursuit of knowledge, to build up the structure, have trusted too much to their own righteousness and invention.

One (Cardinal de Cusa), who, according to Montaclu, was not a doctor, we are told, as quoted above, "* * "tried an ingenious method; he rolled a circle on a "plane, or line, and supposing that its circumference "was applied to it wholly until the point which had "first touched it, touched it again, he therefore justly "inferred that this line would be equal to the circum- "ference. He even conceived the outline of the curve "which the point that first touched the straight line "was to describe which formed the curve since called "the Cycloid." Here, then, is an admission—a saving admission—that if a circle be rolled on a right line until the point of the circle at first in contact with the right line comes again into the right line, the distance between the first and second positions of the point in this right line will equal the circle's circumference. If this were not so the cycloid would have no meaning. Its base must equal the circumference of the generating circle and has been accepted to do so by the doctors ever since Huygens announced his discovery of the cycloid, which, we see, was not his discovery at all, but De Cusa's. The cycloid's base is the measure of π, is so accepted and is so used by all mathematicians. To deny it would be equivalent to rejecting the sacred 3.14159+. Yet there is no connection between them beyond the fact that the doctors give to this base the value 3.14159+, for which there is no authority deducible from the cycloid itself, or its equation. Such being the fact, and 3.14159+ agreeing with the limits of Archimedes, the 3.14159+ has crystallized into an entity as real and as indestructible as some of the phantoms that controlled the decisions of the Roman Inquisitors!

Since the doctors admit that the cycloid's base is the true length of π, though they have not discovered what the length of this base is, we may without offense to them, and thus retain our credit for sanity, try to determine it in the natural way, begun by De Cusa and followed by Huygens. The curves of the geometrical figures used in what follows are Nature's, not mine. They are the actual curves described by a point of one circle rolling around another and fixed one, and a point in the surface of a straight edge around the same fixed circle—the first describing a particular epicycloid, called by geometricians a "Cardioid;" the second the involute of the fixed circle. To make sure of the accuracy of these curves I went to the expense of procuring gears and a rack, and had them suitably arranged with the machinery necessary to describe the curves with the exactness required to show whether the deductions of the mathematical work agree with the graphic relations shown by the figures themselves; but the mathematical work did not precede—did not anticipate—the geometrical figures. The figures were first used as a study to ascertain whether Nature had established any relations between the curves mentioned by which the whole, or a portion, of a circle's circumference could be accurately determined and from a study of the natural curves the relations on which the mathematical work depends were discovered. Therefore the whole result is simply an interpretation of Nature and not my own invention.

In Fig. I, "A" is a fixed circle, and if "B," its equal, be placed in contact with A at the point o, at the extremity of the diameter $o\ n$, and rolled around it in the same plane until the first point, d, of contact, arrives at the position shown in the figure; that is, after the circle B has revolved 45° in contact with A, the point, *d*, will have described on the fixed plane of the circle A the curve *o d*, and if the revolution of B be continued until its point *d* arrives at the starting point, *o*, the symmetrical curve *o.d.v.m.w.o.* will have been described. This curve is the epicycloidal curve called "The Cardioid," and resembles, if it be not the actual curve, the "caustic curve" of optics. Now, if we conceive a straight line, tangent to the two circles at the starting point, *o*, and in the plane of the circles, to move with the rolling circle, so that the line shall be between the rolling and fixed circles, always tangent to them at the point of contact, and at the same time be rolled, or wrapped about the fixed circle, thus bringing consecutively, beginning with the first point of tangency, its points into contact and tangency with the two circles, it is evident that the point of the line, common with the two circles at *o*, at first in contact with the circle A, will describe the curve, called the involute of the circle A, and for the position of B, shown in the figure, the curve *o a* is the involute, and the development of the arc, *o s*,$=45°$ of the circle A; and if the revolution as described of the circle and line be continued half-way around the fixed circle A; that is, until B arrive at the position *n m*, shown in dotted lines, and the line *a s* at the position *r n*, perpendicular to *o n*, the curve *o a p'* will have been described, which is the involute of *o s n*, half of the circle A. Now, if during the revolution of the circle B and the line *a s*, the line *a s* carry with it a plane of which it, the line itself, is a fixed part, and the plane of the rolling circle B be conceived to be in this plane while it rolls on the line *a s*, it is evident that the point *d* of this circle will describe on the said moving plane the cycloid, just the same as if it were rolling on a line in a fixed plane, while at the same time the same point *d* describes on the fixed plane of the circle A the cardioid. These two curves therefore being described by the same point at the same time must be tangent to each other and have the same radius of curvature, as the figure shows. Thus, when the circle B has rolled one-fourth of the way around A, and the line *a s*, carrying with it a plane as stated, has been, under the rolling circle, wrapped tangentially around this same quarter of A, the point, *d*, of the circle B, which was common with the points *a* and *o* at the beginning of the movement, will have described on the moving plane the arc *a d* of the cycloid, tangent to the arc *o d* of the cardioid at the point *d*, described by this point at the same time on the fixed plane of the circle A, and these two arcs will have the same radius of curvature, *d n*. If the movement be continued until the moving circle arrive at the position *n m*, shown in dotted lines, and the line *a s* at the corresponding position *r n*, half the cardioid *o.d.v.m.* and half the cycloid *r m* will have been completed tangent to each other at the point *m* and having the same radius of curvature *o m*. In the differential calculus it is demonstrated that the radius of curvature of the cycloid is twice the distance from the point of contact of the circle with the right line on which it is rolled, to the generating point of the circle; that is, in the figure we are considering, the radius of curvature at the point *d*, the generating point, is twice the distance from it to the point *s* of the line *a s*, on which it is rolling. A property of the cardioid, which this figure clearly shows, enables us to prove this and discovers the close relation between it and the cycloid. The figure shows a dotted circle, *q.d.n.t.*, twice as large as the circle, A. This circle, it will be observed, passes through the point *d* of the moving circle B, and *n* of the fixed circle A, its center being at *o*. This larger circle, it is evident, can be rolled,—wrapped,—around the fixed one, its center at *o*, moving constantly in the fixed circle A, the point of contact or tangency, *n*, being always directly opposite, at the extremity of a diameter of A. Now, any point of this larger circle, in such a movement, will describe a cardioid, the point *d*, for instance, common to it and the circle B, describes the cardioid shown in the figure. In other words, the cardioid is susceptible of being described in two ways—one by a point of any circle rolling on and around an equal fixed circle; the other, by any point of a circle twice as great as the fixed one rolling around and tangent to it, the fixed circle being inside of the larger moving one. When the two methods are combined in one movement the center of the greater travels around the center of the fixed circle twice as fast as the center of the smaller moving circle. This combined movement is shown in the figure, the generating point *d* being common to the moving circles. To repeat, this property of the cardioid of being susceptible of two ways of generation, enables us to prove the deduction of the mathematics, that the radius of curvature of the cycloid is twice the distance from the generating point of the circle to the point of its contact with the line on which it is rolling. It is evident that the point *s* is not the center for the radius of curvature of the point *d* of the cardioid, since the same point may be generated from a center at *n*, and for the same reason it is not a center of curvature for the corresponding point *d* of the cycloid, since this latter is tangent to the cardioid at the point *d*, and has the same radius of curvature. Now, the distance *d n*, being the sum of two equal parts, is twice as great as *d s*, one of these parts; and the same is true for every point of the cardioid and cycloid, as they are successively generated by the common point, *d*. Such being the intimate relations between the cardioid and cycloid, it would seem but natural to seek for some relation between the former and the involute of the circle, also generated by the same movement, by which the length of a part, or the whole, of a circle's circumference could be accurately determined.

The equation of the cardioid, referred to rectangular axes, may be found in the mathematical books, but the following of simpler form is readily deducible, viz.:

$$x^2 + y^2 + 2Rx - 2R\sqrt{x^2 + y^2} = 0$$

An equation of the 4th degree, x and y being the variables, R the radius of the fixed circle, and the origin of co-ordinates at the origin—the cusp point—of the curve, abscissæ measured to the left of the origin being positive.

In what follows the *unit* is the diameter of the fixed circle A; the radius R($o\ c$) therefore is ½.

In Fig. II, $o\ x$ and $o\ y$ are the rectangular co-ordinate axes of abscissæ and ordinates respectively. For the point d, $x = o\ b$ and $y = d\ b$. This figure shows one position of the revolving circle B, and the line $a\ s\ e$, in the generation of the cardioid and involute of the fixed circle A. It is evident that the arc, $o\ s$, of the fixed circle is equal to the arc, $d\ s$, of the revolving circle, since every point of $d\ s$ has been in contact with every corresponding point of $o\ s$—the circles being equal, and the point d having started from the point o. So, for every position of the revolving circle, the distance from the generating point d to the point of contact of the two circles is—must be—equal to the distance from the point o of starting to the point of contact of the two circles. Therefore in the figure the line $d\ o$, connecting the generating point d with its initial position o, is parallel with the line $c\ c'$ connecting the centers of the two circles, and in like manner must the line connecting the generating point with its initial position, in any position of the two circles, be parallel with the line connecting their centers. Now, if the point r, where the line $o\ y$, which is tangent to the fixed circle at o, and the line $a\ s$, the development of the arc $o\ s$, tangent to the circles at s, intersect, be connected by the lines $r\ d$ and $r\ o$; also the generating point a of the involute be connected with the points d and o by the lines $a\ d$ and $a\ o$, there will be formed the irregular four-sided figure $a.d.r.o.$ divided equally by the line $a\ r$, and unequally by the line $d\ o$, having the sides $r\ d$ and $r\ o$ equal and tangent respectively to the circles B and A at the points d and o; also having the sides $a\ d$ and $a\ o$ equal. It is evident that like figures can be formed for all positions of the two curves (fig. I) $o\ a$ and $o\ d$ for a revolution of 45°, but there will be one exception to this equality, in pairs, of the sides of the figures; also for a part of the generation of the curves, the pair $a\ d$ and $a\ o$ will be less than the pair $r\ d$ and $r\ o$, and for the other part the reverse will obtain. Therefore, at the point where the less passes to the greater, and vice versa, there must be equality between them all. The line *d a*, in fig. II, if produced downward until it intersects the axis *o x* will make, on the left, an acute angle with it, and conceiving the circle B rolled back toward the initial position, this angle grows less and less as the point *a* approaches the point *o*, until it becomes about zero, about the time the line *a d* itself becomes zero—the two lines *a d* and *a o* decreasing in the movement more rapidly than the lines *r d* and *r o*, and the point *a* remaining inside, to the right of, the ordinate *d b* of the point *d*. Reversing the movement, upward, the angle referred to becomes greater and greater and the lines *a d* and *a o* increase in length more rapidly than *r d* and *r o*, the point *a* gradually approaching the ordinate *d b*, until it passes into and becomes a point of this ordinate when the angle becomes 90°. Now, the line *a d* has always a point, at *d*, in common with this ordinate, and when the point *a* passes into it, the line *a d* becomes a part of the ordinate. Arrest the movement at the instant this takes place, and we will have the position of all the right lines and curves shown in fig. III. That is, we will have the position of these right lines and curves at the instant that the inequality in the pairs of lines *r d—r o* and *a d—a o*, reverses, as above explained. To pass from one state of inequality to the opposite they must necessarily pass through the state of equality for all of the four lines. That this is true, as graphically shown by the figure to be so, it is easily demonstrated as follows. When the point *a* passes into the ordinate *d b*, *a d* becomes a part of that ordinate, and, since *d b*, the ordinate, and *o r* are perpendicular to the axis *o x*, *a d* is parallel to *o r*. The right line *d o*, intersecting these parallels at *d* and *o*, makes equal angles with them ; that is, the angle *d o r* equals the angle *a d o*, but the angle *d o r* equals *o d r*, therefore the angle *a d o*, its equal, equals *o d r*. It follows also, since *a d o* equals *a o d*, that *a o d* equals *d o r* ; in other words, that the four angles made by the line *d o* with the four sides of the figure *a.o.r.d.* at *d* and *o* are equal, which could not be so, unless the line *d o* bisected the line *a r*, the other axis of the figure, making the four lines *a o*, *o r*, *r d*, and *d a* all equal and opposite sides parallel to each other. Now, when these lines are equal, which we see they are both graphically and mathematically, the moment the point *a* passes into the ordinate *d b* of the point *d* of the cardioid, besides the parallelism and equality just demonstrated, something else takes place that accurately determines the angle *o c s* of the arc *o s*, of which the right line *a s* is the development. This something is, that the radius vector of the point *a*, of the involute *o a p'*, makes an angle of 45° with the co-ordinate axes *o x* and *o y*.

There are a number of ways of proving this; what appears to be the simplest and most direct is chosen, as follows:

Draw (fig. III) the auxiliary lines, *t p* perpendicular to *d r* and *a h* to *o r*. It is evident that the two right angle triangles *t a b* and *d a p* have their angles respectively equal—the angle *t a b* being equal to *d a p*, and *a t b* equal to *a d p*. It is also evident that the line *a p* equals the line *a h*, for the lines *d r* and *o r* make equal angles with the line *a r*, from the point *a* of which, the lines *a p* and *a h* are drawn perpendicular to the lines *d r* and *o r* respectively; also, for the same reason, the line *d p* equals *o h*, which latter by construction equals *a b*. Hence $d p = a b$; therefore the said right angle triangles have, besides equal angles, each a side equal to a side of the other and consequently the remaining sides are respectively equal,—*d a* being equal to *t a*, and *t b* being equal to *a p*. But $d a = a o$, therefore $t a = a o$, and *t a o* being, by construction, a right angle, the triangle *t a o* is isosceles, and the line *a b* passing through the right angle perpendicular to the side *t o*, bisects it and the right angle, making $t b = b o$, hence $t a b = b a o = 45°$; consequently *a o b* and *a o h* each = 45°, and *b.a.h.o.* is a square of which *a o* is a diagonal. As previously demonstrated, the line *d o* is parallel to *c c'* and bisects the angles at *d* and *o*, making the four angles for the condition existing shown in fig. III, equal each to 22°.5, therefore the angle $t\,o\,d$, which $= o\,c\,s, = a\,o\,b + a\,o\,d = 45^\circ + 22^\circ.5 = 67^\circ.5$. Thus the line $a\,s$ is the development of, and equals the arc, $o\,s$, of 67°.5. The length of this line, $a\,s$, can now be determined by the geometrical relations, which become apparent by examination of the figure, but not as a whole, as these relations do not embrace it as such, but in parts. The line is equal to $a\,r + r\,s$, also to $a\,z + z\,s$; the value of which parts can be determined separately and when added respectively will give the whole. The figure shows that $r\,s$ (which $= r\,o$) is the tangent of half the arc $o\,s$, which equals 33°45′, and that $a\,r$ is equal to $2\,a\,z$ The determination therefore of the value of $a\,z$, determines that of $a\,r$. Now, $a\,z = a\,d \times \sin a\,d\,z$, and to find the value of $a\,d$ in this equation, that of $d\,z$ must be used, which is determined as follows: $d\,z$ (which $= z\,o$) equals $q\,s$ which equals $\frac{1 - \cos \text{ of } o\,c\,s}{2}$. We know $o\,c\,s = 67^\circ.5$; therefore $d\,z = \frac{1 - \cos 67^\circ.5}{2}$: also $d\,z = a\,d \times \cos a\,d\,z = a\,d \times \cos 22^\circ.5$. From these two values of $d\,z$, we obtain the value of $a\,d = \frac{1 - \cos 67^\circ.5}{2 \cos 22^\circ.5}$, which substituted in the equation above gives

$$a\,z = \frac{1 - \cos 67^\circ.5}{2 \cos 22^\circ.5} \times \sin 22^\circ.5.$$

Therefore $2\,a\,z = a\,r = \frac{(1 - \cos 67^\circ.5)}{\cos 22^\circ.5} \times \sin 22^\circ.5$

Hence the sum of the parts

$$r\,s + a\,r = \tfrac{1}{2} \operatorname{Tan} 33^\circ 45' + \frac{(1 - \cos 67^\circ.5)}{\cos 22^\circ.5} \times \sin 22.5.$$

Or since $\frac{\sin 22^\circ.5}{\cos 22^\circ.5} = \operatorname{Tan} 22^\circ.5$,

$$r\,s + a\,r = \tfrac{1}{2} \operatorname{Tan} 33^\circ 45' + (1 - \cos 67^\circ.5) \times \tan 22^\circ.5 \quad (1)$$

For the sum of the parts $a\,z$ and $z\,s$, for which values can also be obtained from the geometrical relations, we have, as just determined, the value of $a\,z$, and $z\,s = o\,q$; but $o\,q$ is equal to $o\,c \times \sin$ of $o\,c\,s$, in which $o\,c = R = \tfrac{1}{2}$, and, as has been shown, $o\,c\,s = 67^\circ.5$. Therefore $z\,s = \tfrac{1}{2} \times \sin 67^\circ.5 = \frac{\sin 67^\circ.5}{2}$, consequently

$$z\,s + a\,z = \frac{\sin 67^\circ.5}{2} + \frac{1 - \cos 67^\circ.5}{2 \cos 22^\circ.5} \times \sin 22^\circ.5$$

or since $\frac{\sin 22^\circ.5}{\cos 22^\circ.5} = \text{Tan } 22^\circ.5$,

$$z\,s + a\,z = \frac{\sin 67^\circ.5}{2} + \frac{(1 - \cos 67^\circ.5)}{2} \times \text{Tan } 22^\circ.5 \qquad (2)$$

Now, if in equations "(1)" and "(2)" we substitute the values of the trigonometrical functions

we obtain, $a\,r + r\,s = a\,s = 0.5897899+$

and $a\,z + z\,s = a\,s = 0.58979005+$

Again, the figure shows that the development

$$r\,s + a\,r = \tfrac{1}{2}\text{Tan } 33^\circ 45' + (\text{Tan } 33^\circ 45' \times \sin 22^\circ.5)$$

from which we obtain

$$a\,s = a\,r + r\,s = 0.5897896+$$

These values are the same for four places of decimals, two of them for six places, but they are all too small, and the reason they differ is due to the cause that makes them too small, viz.: the value of these functions depends on the square roots of imperfect squares. Now, in the development of the circle, the increase is constant; that is, the development of any part is equal to that of any other of an equal number of degrees. So this value of the line $a\,s$ is to $67^\circ.5$ as the whole development is to 360°. Therefore, taking the value of $a\,s$ to be the four decimals common to the three values just found,

$67^\circ.5 : 0.5897 :: 180^\circ : r\,n$ (fig. I)

$r\,n = 1.5725+$

and $2\,r\,n = \pi = 3.1450+$, which, for reasons stated, though greater than $3.141592+$, is yet too small.

Apparently there are other geometrical relations that may be established by further study of the figures I have used in this paper, that may throw additional light on this interesting subject. For the present all that remains to be said is, is this "A Last Word," or the last word, for the Quadrateurs?

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✕ May be too large — the
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Fig. 1

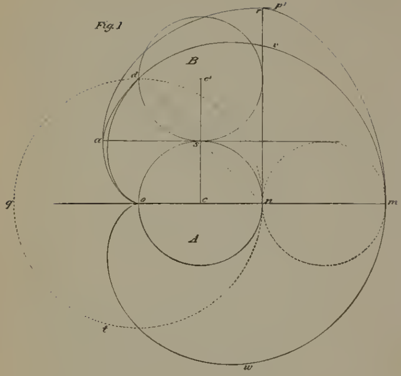


Fig. II.

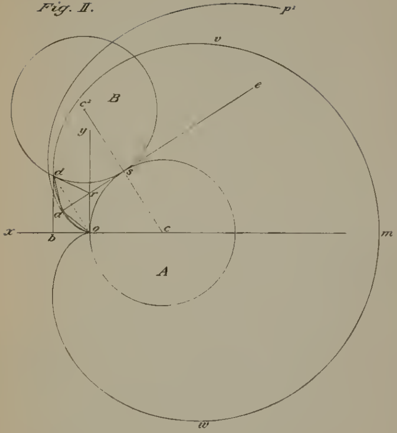
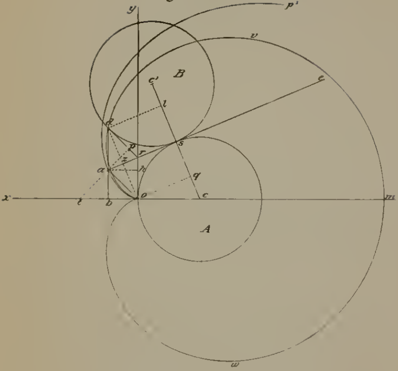


Fig. III



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